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Evolution of SU(1, 1) coherent states in harmonic oscillators with time-dependent masses

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Abstract. We study the evolution of SU(1, 1) coherent states, which may represent squeezed vacuum states, in harmonic oscillators with time-dependent masses. Such oscillators may model a quantised electromagnetic field in a cavity where the field changes with time under the action of some reservoir. We consider three types of variable mass: a decaying mass, a pulsating mass, and a mass with damping and pulsation. We have found that in the most interesting cases involving pulsation, a high degree of squeezing (enhanced squeezing) may be obtained in the vicinity of the resonance where the pulsation frequency is twice the natural oscillator frequency.

1. Introduction

Recently, there has been much interest in the quantum mechanical description of harmonic oscillators with time-dependent masses. Of particular relevance to the present work is a series of papers by Colegrave and Abdalla [1-6], Abdalla [7, 8] and Abdalla and Ramjit [9]. These authors have considered the time-dependent mass problem for the harmonic oscillator in the context of a quantised electromagnetic field in a Fabry-Pérot cavity where the time-dependent mass can model a decaying or driven cavity.

In the present paper, we shall reconsider the problem of harmonic oscillators with time dependent masses from the point of view of the generalised coherent states (CS) associated with the dynamical group SU(1, 1) [10]. Specifically, we show that the Hamiltonians associated with such systems may be cast into a form linear in the generators of SU(1, 1) which implies that an arbitrary initial SU(1, 1) CS is preserved under time evolution by such a system [11]†. Moreover we shall study the evolution of SU(1, 1) CS for particular mass laws, namely a decaying mass: $M(t)=M_0\,e^{-2\gamma t}$, and oscillating mass: $M(t)=M_0(1+\alpha\sin\lambda t)(|\alpha|<1)$, and the case of a mass law with damping and pulsation: $M(t)=M_0\exp[2(\gamma t+\mu\sin\nu t)]$. These mass laws may represent phenomenologically the interaction of the cavity field with the cavity walls in the case of damping in a finite Q cavity or with a reservoir of two-level atoms absorbing and re-emitting photons causing oscillations in the field (see below). Now the SU(1, 1) CS may be taken as a specific type of quantum state having no classical analogue, namely a squeezed state or, more specifically, a squeezed vacuum state [12, 13] the single mode quantised electromagnetic field. We therefore retain this picture of the

† Equation (2.25) in [11] is incorrect. The correct equation is (2.19) of the present paper.

harmonic oscillator with a variable mass as a model for a quantised electromagnetic field in a cavity. We shall study the effects of the cavity on an injected SU(1, 1) CS (which may be a squeezed vacuum state). A specific case of an initial state would of course be just the usual unsqueezed vacuum from which a squeezed state would be expected to arise. Indeed, Hong-Yi and Zaidi [14] have shown that squeezing can arise by the change of mass of a harmonic oscillator. (Squeezing can also arise by a time dependent frequency [15].)

The plan of the paper is as follows. In section 2 we review the quantised cavity field with a time dependent mass parameter and relate that formalism to the dynamical group SU(1, 1). We give only a brief review of the SU(1, 1) CS but we do present the general method of solving the time evolution of an initial SU(1, 1) CS for our particular problem. This basically involves solving a first-order differential equation of a Riccati type for a complex parameter representing the phase space of the 'classical' motion. In section 3 we study the particular mass laws previously given. A few brief remarks are made in the conclusion of section 4.

Before going to the next section however, we point out that a related study has already been carried out for a damped harmonic oscillator [16]. The interpretation in that case is somewhat different than in the present case where the mass carries the time dependence. In [16] we used the Kanoi-Caldirola [17] (KC) Hamiltonian for which recently Yeon *et al* [18] and Oh *et al* [19] have constructed exact coherent states which are generalisations of the usual harmonic oscillator coherent states. For critiques of the use of the KC Hamiltonian see Greenberger [20] and Dekker [21].

2. SU(1, 1) formulation of a harmonic oscillator with a time-dependent mass

In this section we reformulate the treatment of Colegrave and Abdalla [1-6] of the harmonic oscillator of variable mass by using the SU(1, 1) dynamical group. We begin, as a matter of motivation, by considering the quantised electromagnetic field with a time dependent mass parameter, essentially along the lines given in references [1] and [4]. Aside from giving the general setting for the problem, this allows us to properly introduce the quadrature operators for the field with the variable mass parameter.

Following Sargent *et al* [22] and Colegrave and Abdalla [1, 4], we write the non-vanishing component of the electric field as

$$\mathscr{E}_x(z, t) = q(t)\left(\frac{2\omega_0^2 M(t)}{V\varepsilon_0}\right)^{1/2} \sin Kz \tag{2.1}$$

where V is the volume of the cavity, $M(t)$ is a time-dependent parameter with the dimensions of mass, and $q(t)$ has the dimensions of length. From Maxwell's equations $\nabla \times \mathscr{H} = \varepsilon_0\, \partial\mathscr{E}/\partial t$, the magnetic field is

$$\mathscr{H}_y(z, t) = [2\omega_0^2\varepsilon_0/VK^2]^{1/2} M^{-1/2}(t)\, p(t) \cos Kz \tag{2.2}$$

where

$$p(t) = M^{1/2}(t)\frac{\mathrm{d}}{\mathrm{d}t}[M^{1/2}(t) q(t)] \tag{2.3}$$

in the canonical momentum conjugate to the coordinate $q(t)$. The Hamiltonian of the system is just

$$H = \frac{1}{2}\int_V (\varepsilon_0 \mathscr{E}_x^2 + \mu_0 \mathscr{H}_y^2)\,\mathrm{d}\tau$$
$$= \frac{p^2}{2M(t)} + \tfrac{1}{2}M(t)\omega_0^2 q^2. \tag{2.4}$$

Quantisation proceeds by demanding that $[q, p] = \mathrm{i}$.

We now introduce annihilation and creation operators by defining

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(\sqrt{M_0\omega_0}\,q + \mathrm{i}\frac{p}{\sqrt{M_0\omega_0}}\right) \\ a^+ &= \frac{1}{\sqrt{2}}\left(\sqrt{M_0\omega_0}\,q - \mathrm{i}\frac{p}{\sqrt{M_0\omega_0}}\right) \end{aligned} \tag{2.5}$$

such that $[a, a^+] = 1$. The mass M_0 is the constant 'reference mass' of the harmonic oscillator in the case of no time dependence. In terms of these operators the cavity fields become

$$\mathscr{E}_x(z, t) = \left(\frac{\omega_0}{V\varepsilon_0}\right)^{1/2}\left(\frac{M(t)}{M_0}\right)^{1/2}(a + a^+)\sin Kz \tag{2.6}$$

$$\mathscr{H}_y(z, t) = \left(\frac{\omega_0^3\varepsilon_0}{VK^2}\right)^{1/2}\frac{1}{\mathrm{i}}\left(\frac{M_0}{M(t)}\right)^{1/2}(a - a^+)\cos Kz. \tag{2.7}$$

We introduce the field quadratures (see the review in [23]) as

$$\begin{aligned} X_1 &= \left(\frac{M(t)}{M_0}\right)^{1/2} X_{10} \\ X_2 &= \left(\frac{M_0}{M(t)}\right)^{1/2} X_{20} \end{aligned} \tag{2.8}$$

where X_{10} and X_{20} are the quadrature operators for mass M_0:

$$\begin{aligned} X_{10} &= \tfrac{1}{2}(a + a^+) \\ X_{20} &= \frac{1}{2\mathrm{i}}(a - a^+). \end{aligned} \tag{2.9}$$

The commutation relations for the quadrature operators are

$$[X_1, X_2] = [X_{10}, X_{20}] = \mathrm{i}/2 \tag{2.10}$$

which implies the uncertainty relations

$$\begin{aligned} &\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geq \tfrac{1}{16} \\ &\langle(\Delta X_{10})^2\rangle\langle(\Delta X_{20})^2\rangle \geq \tfrac{1}{16}. \end{aligned} \tag{2.11}$$

Squeezing is said to exist if for any quadrature operator X, $\langle(\Delta X)^2\rangle < \frac{1}{4}$. In what follows we shall perform a scale transformation such that we need only consider the operators X_{10} and X_{20}.

We now introduce the su(1, 1) Lie algebra which consist of the generators K_0, K_1, and K_2 satisfying the commutation relations

$$[K_1, K_2] = -\mathrm{i}K_0 \qquad [K_2, K_0] = \mathrm{i}K_1 \qquad [K_0, K_1] = \mathrm{i}K_2 \tag{2.12}$$

or with $K_\pm = K_1 \pm \mathrm{i}K_2$,

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0. \tag{2.13}$$

The algebra may be realised in terms of the annihilation and creation operators as

$$\begin{aligned} K_0 &= \tfrac{1}{4}(a^+a + aa^+) = \tfrac{1}{2}(a^+a + \tfrac{1}{2}) \\ K_+ &= \tfrac{1}{2}(a^+)^2 \qquad K_- = \tfrac{1}{2}a^2. \end{aligned} \tag{2.14}$$

The relevant unitary irreducible representations are the positive discrete series $\mathscr{D}^+(k)$ whose basis we denote as $\{|m, k\rangle\}$ for $k>0$, $m = 0, 1, 2, \ldots$. The number k is the Bargmann index and is related to the Casimir operator $C = K_0^2 - K_1^2 - K_2^2$ according to

$$C|m, k\rangle = k(k-1)|m, k\rangle. \tag{2.15}$$

For the realisation of equations (2.14), one has $C = -\frac{3}{16}$ such that $k = \frac{1}{4}$ or $\frac{3}{4}$. The Fock space of photon number states $|n\rangle$, such that $N|n\rangle = n|n\rangle$ where $N = a^+a$, is split into even number states $k = \frac{1}{4}$, and odd number states $k = \frac{3}{4}$. That is, the SU(1, 1) group state $|m, \frac{1}{4}\rangle$ corresponds to photon states $|2m\rangle$ while the state $|m, \frac{3}{4}\rangle$ corresponds to $|2m+1\rangle$, which is clear since $K_0 = \frac{1}{2}(N + \frac{1}{2})$.

The SU(1, 1) CS are defined as [10]

$$|\xi, k\rangle = \exp(\alpha K_+ - \alpha^* K_-)|0, k\rangle \tag{2.16}$$

where $\alpha = -\frac{1}{2}\theta\, \mathrm{e}^{-\mathrm{i}\phi}$ and θ and ϕ are group parameters with the ranges $0 < \theta < \infty$, $0 \leqslant \phi \leqslant 2\pi$ and $\xi = -\tanh(\frac{1}{2}\theta)\, \mathrm{e}^{-\mathrm{i}\phi}$. These states may be expanded in the basis $|m, k\rangle$ as [10]

$$|\xi, k\rangle = (1 - |\xi|^2)^k \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+2k)}{m!\,\Gamma(2k)}\right)^{1/2} \xi^m |m, k\rangle. \tag{2.17}$$

Note that in our case, the CS are linear superpositions of all even ($k = \frac{1}{4}$) or all odd ($k = \frac{3}{4}$) photon states. In the rest of the paper we consider only the CS for $k = \frac{1}{4}$, which includes the vacuum. The complex parameter ξ, where $|\xi| < 1$, defines the classical phase space of the SU(1, 1) CS, which has been shown to be in the form of the Lobachevsky plane [10]. It has previously been shown that for a Hamiltonian consisting of a linear combination of the SU(1, 1) generators, e.g.

$$H(t) = A(t)K_0 + f(t)K_+ + f^*(t)K_- + B(t) \tag{2.18}$$

where $A(t)$ and $B(t)$ are real and $f(t)$ is complex, but otherwise arbitrary, preserves the coherence of an initial SU(1, 1) CS, $|\xi, k\rangle$, under time evolution and that ξ satisfies the classical equation of motion

$$\dot{\xi} = -\mathrm{i}A(t)\xi - \mathrm{i}f^*(t)\xi^2 - \mathrm{i}f(t). \tag{2.19}$$

In the next section, we shall present solutions (mostly numerical) for this equation upon specifying $A(t)$ and $f(t)$.

Using the annihilation and creation operators of (2.5), the SU(1, 1) generators may be written as

$$
\begin{aligned}
K_0 &= \tfrac{1}{2}(a^+a+\tfrac{1}{2}) = \frac{1}{4}\left(\frac{p^2}{M_0\omega_0}+M_0\omega_0 q^2\right)\\
K_1 &= \tfrac{1}{4}(a^{+2}+a^2) = \frac{1}{4}\left(M_0\omega_0 q^2-\frac{p^2}{M_0\omega_0}\right)\\
K_2 &= \frac{1}{4\mathrm{i}}(a^{+2}-a^2) = -\tfrac{1}{4}(qp+pq).
\end{aligned}
\tag{2.20}
$$

From these we have that

$$
\begin{aligned}
q^2 &= \frac{2}{M_0\omega_0}(K_0+K_1)\\
p^2 &= 2M_0\omega_0(K_0-K_1)
\end{aligned}
\tag{2.21}
$$

so that the Hamiltonian of (2.4) becomes

$$
H=\omega_0\left[\frac{M_0}{M(t)}(K_0-K_1)+\frac{M(t)}{M_0}(K_0+K_1)\right].
\tag{2.22}
$$

This is, of course, of the form of (2.18) with

$$
\begin{aligned}
A(t) &= \omega_0\left(\frac{M(t)}{M_0}+\frac{M_0}{M(t)}\right)\\
f(t) &= \frac{\omega_0}{2}\left(\frac{M(t)}{M_0}-\frac{M_0}{M(t)}\right).
\end{aligned}
\tag{2.23}
$$

However, we now perform a time-dependent scale transformation to effectively reduce the coefficient of K_0 to a constant, thus reducing the number of time-dependent functions needed in (2.19). Since our transformation is to be time dependent, we start with the time-dependent Schrödinger equation

$$
H(t)|\psi\rangle = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle.
\tag{2.24}
$$

Let $V(t)$ be a time dependent unitary operator effecting the transformation. The transformed Hamiltonian is then

$$
\bar{H}(t)=V(t)HV^+(t)-\mathrm{i}V(t)\frac{\partial V^+(t)}{\partial t}
\tag{2.25}
$$

such that the Schrödinger equation now reads

$$
\bar{H}(t)\overline{|\psi\rangle}=\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\overline{|\psi\rangle}
\tag{2.26}
$$

where $\overline{|\psi\rangle}=V(t)|\psi\rangle$. Since K_2 of (2.20) is a generator of scale transformations [24], we take $V(t)$ of the form

$$
V_2(t)=\mathrm{e}^{-\mathrm{i}\lambda(t)K_2}
\tag{2.27}
$$

where $\lambda(t)$ is shortly to be determined and where the subscript 2 indicates that $V_2(t)$ is generated by K_2. From the Baker-Hausdorff formula, we have [25]

$$V_2(t)(K_0 \pm K_1) V_2^+(t) = \mathrm{e}^{\pm\lambda(t)}(K_0 \pm K_1). \tag{2.28}$$

Thus with the choice $\lambda(t) = \ln(M_0/M(t))$, we have

$$V_2(t) H(t) V_2^+(t) = 2\omega_0 K_0 \tag{2.29}$$

so that

$$\begin{aligned}\bar{H}(t) &= 2\omega_0 K_0 + \frac{\mathrm{d}\lambda}{\mathrm{d}t} K_2 \\ &= 2\omega_0 K_0 - \frac{1}{M(t)} \frac{\mathrm{d}M}{\mathrm{d}t} K_2 \\ &= 2\omega_0 K_0 + \frac{\mathrm{i}}{2} \frac{1}{M(t)} \frac{\mathrm{d}M(t)}{\mathrm{d}t} (K_+ - K_-).\end{aligned} \tag{2.30}$$

This is of the form of (2.18) with $A = 2\omega_0$ and

$$f(t) = \frac{\mathrm{i}}{2} \frac{1}{M(t)} \frac{\mathrm{d}M(t)}{\mathrm{d}t}. \tag{2.31}$$

The time-dependent scale transformation we have performed here is equivalent to the canonical transformation carried out by Colegrave and Abdalla [1-4].

We now consider the transformation of the variances of the field quadratures. Since it can be shown that

$$V(t)(a \pm a^+) V^+(t) = \mathrm{e}^{\pm\lambda(t)/2}(a \pm a^+) \tag{2.32}$$

that the quadrature operators (2.8) transform to the new representation as

$$\begin{aligned}V(t) X_1 V^+(t) &= X_{10} = \frac{1}{2}(a + a^+) \\ V(t) X_2 V^+(t) &= X_{20} = \frac{1}{2\mathrm{i}}(a - a^+).\end{aligned} \tag{2.33}$$

Since we are dealing with superpositions of only even photon states, we shall always have $\langle X_{10} \rangle = 0 = \langle X_{20} \rangle$. Then from (2.14), the variances of X_{10} and X_{20} are given as [12, 13]

$$\begin{aligned}\langle (\Delta X_{10})^2 \rangle &= \langle K_0 + K_1 \rangle \\ \langle (\Delta X_{20})^2 \rangle &= \langle K_0 - K_1 \rangle.\end{aligned} \tag{2.34}$$

For an SU(1, 1) cs, $|\xi, k\rangle$, these variances are given as [12, 13]

$$\begin{aligned}\langle (\Delta X_{10})^2 \rangle &= k\left(\frac{1+|\xi|^2}{1-|\xi|^2}\right) + \frac{2k \operatorname{Re}(\xi)}{1-|\xi|^2} \\ \langle (\Delta X_{20})^2 \rangle &= k\left(\frac{1+|\xi|^2}{1-|\xi|^2}\right) - \frac{2k \operatorname{Re}(\xi)}{1-|\xi|^2}\end{aligned} \tag{2.35}$$

where, for the squeezed vacuum, k is set equal to $\frac{1}{4}$. For a discussion of the squeezing associated with an SU(1, 1) cs, see [12] and [13].

As a final remark in this section, we discuss the relation between a SU(1, 1) CS driven by $\bar{H}(t)$ and one driven by $H(t)$. Let $|\xi, k\rangle$ be, up to a time-dependent phase factor, the CS driven by $\bar{H}(t)$ and $|\xi', k\rangle$ be that driven by $H(t)$. The two states are related (up to a factor) by

$$|\xi', k\rangle = V_2^+(t)\overline{|\xi, k\rangle} \tag{2.36}$$

where $V_2^+(t) = e^{i\lambda(t)K_2}$ is a finite SU(1, 1) group transformation. The group element in the 2×2 non-unitary representation is determined by realising K_2 as $K_2 = -i\sigma_1/2$ where σ_1 is a Pauli matrix and K_2 is now non-Hermitian. The unitary infinite-dimensional representations are obtained from the 2×2 non-unitary representation by induction [26]. Thus we have, upon expanding the exponential,

$$V_2^+(t)_{(2\times 2)} = e^{i\lambda(t)K_2}_{(2\times 2)} = \begin{pmatrix} \cosh\frac{1}{2}\lambda(t) & \sinh\frac{1}{2}\lambda(t) \\ \sinh\frac{1}{2}\lambda(t) & \cosh\frac{1}{2}\lambda(t) \end{pmatrix}. \tag{2.37}$$

Now for any SU(1, 1) transformation $T(g)$, where g is the associated 2×2 group element

$$g = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \qquad |a|^2 + |b|^2 = 1 \tag{2.38}$$

it can be shown that [27]†

$$T(g)|\xi, k\rangle = e^{-i\Phi}|\xi', k\rangle \tag{2.39}$$

where

$$\xi' = \frac{a\xi + b}{b^*\xi + a^*} \qquad \Phi = 2k \arg(a^* + b^*\xi). \tag{2.40}$$

Thus, from (2.36), (2.37) and (2.40), we have

$$\xi'(t) = \frac{\xi(t)\cosh\frac{1}{2}\lambda(t) + \sinh\frac{1}{2}\lambda(t)}{\xi(t)\sinh\frac{1}{2}\lambda(t) + \cosh\frac{1}{2}\lambda(t)}. \tag{2.41}$$

If $\xi(t)$ is a solution of (2.19) for Hamiltonian $\bar{H}(t)$, then $\xi'(t)$ is a solution for Hamiltonian $H(t)$.

To calculate the average energy of the cavity $E(t)$ we must use $H(t)$ of (2.4) and the state $|\xi'(t), k\rangle$ such that

$$E(t) = \langle \xi'(t), k|H(t)|\xi'(t), k\rangle. \tag{2.42}$$

In what follows, we shall mainly work in the transformed picture and we will not consider the energy further.

3. Applications

We now turn to the applications for the various mass laws mentioned in the introduction.

3.1. Exponentially decaying mass

We assume that the mass varies with time according to

$$M(t) = M_0 e^{-2\gamma t}. \tag{3.1}$$

† One must replace ξ by $-\xi$ in [17] to agree with our convention.

This mass law describes the damping of the field as it leaks through the cavity wall if $\gamma=\omega_0/2Q$ where ω_0 is the frequency of the decaying mode and Q is the quality factor of the cavity [1]. This form of the time-dependent mass gives rise to the phenomenological damping term $\omega_0\mathscr{E}/Q$ in the semiclassical equations for the field in the work of Jaynes and Cummings [28]. In this case the Hamiltonian $H(t)$ becomes

$$H(t)=2\omega_0[K_0\cosh(2\gamma t)-\tfrac{1}{2}\sinh(2\gamma t)(K_++K_-)] \tag{3.2}$$

while $\bar{H}(t)$ becomes

$$\begin{aligned}\bar{H}&=2\omega_0K_0+2\gamma K_2\\&=2\omega_0K_0-\mathrm{i}\gamma(K_+-K_-)\end{aligned} \tag{3.3}$$

which is independent of time. Under certain conditions, namely when $\gamma<\omega_0$, this can be simplified even further by performing a transformation to remove the non-compact generator K_2. This is accomplished by using a unitary transformation generated by K_1,

$$V_1=\mathrm{e}^{-\mathrm{i}\beta K_1} \tag{3.4}$$

and defining

$$\tilde{H}=V_1\bar{H}V_1^+=K_0[2\omega_0\cosh\beta-2\gamma\sinh\beta]+K_2[2\gamma\cosh\beta-2\omega_0\sinh\beta] \tag{3.5}$$

where we have used the Baker–Hausdorff formulae

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\beta K_1}K_0\,\mathrm{e}^{\mathrm{i}\beta K_1}&=K_0\cosh\beta-K_2\sinh\beta\\ \mathrm{e}^{-\mathrm{i}\beta K_1}K_2\,\mathrm{e}^{\mathrm{i}\beta K_1}&=K_2\cosh\beta-K_0\sinh\beta.\end{aligned} \tag{3.6}$$

With the choice $\beta=\tanh^{-1}(\gamma/\omega_0)$ if $\gamma<\omega_0$, $\tilde{H}$ reduces to

$$\tilde{H}=2\omega K_0 \tag{3.7}$$

where ω is the modified frequency $\omega=(\omega_0^2-\gamma^2)^{1/2}$. Thus, as previously shown by Colegrave and Abdalla [1], the effect of the damping is to modify the frequency of the cavity, as long as the damping is weak ($\omega_0>\gamma$). The eigenstates of the Hamiltonian $\tilde{H}$ are just the SU(1, 1) groups basis $|m,k\rangle$ and the energy eigenvalues are $E_m=2\omega(m+k)$. On the other hand, if $\gamma>\omega_0$, one must remove the K_0 generator and $\tilde{H}$ will have a continuous spectrum. We shall not pursue this case of overdamping.

Now the equation of motion for the Hamiltonian $\tilde{H}$ for an initial SU(1, 1) cs $|\widetilde{\xi(0),k}\rangle$,

$$\dot{\xi}=-2\mathrm{i}\omega\xi \tag{3.8}$$

whose solution is

$$\xi(t)=\xi(0)\,\mathrm{e}^{-2\mathrm{i}\omega t} \tag{3.9}$$

which is just a circle centred at the origin of the ξ-plane. Now for the Hamiltonian $\bar{H}$, the motion easily obtained by the transformation

$$\overline{|\xi'(t),k\rangle}=V_1^+|\widetilde{\xi(t),k}\rangle \tag{3.10}$$

since, from the 2×2 realisation of the su(1, 1) algebra, where $K_1=\mathrm{i}\sigma_2/2$,

$$V^+_{1(2\times2)}=\mathrm{e}^{\mathrm{i}\beta K_1}_{(2\times2)}=\begin{pmatrix}\cosh(\beta/2) & \mathrm{i}\sinh(\beta/2)\\ -\mathrm{i}\sinh(\beta/2) & \cosh(\beta/2)\end{pmatrix} \tag{3.11}$$

then using (2.39) and (2.40), we have

$$\xi'(t)=\frac{\cosh(\beta/2)\xi(t)+\mathrm{i}\sinh(\beta/2)}{-\mathrm{i}\sinh(\beta/2)\xi(t)+\cosh(\beta/2)} \tag{3.12}$$

where $\beta = \tanh^{-1}(\gamma/\omega_0)$. $\xi'(t)$ is the solution of (2.19) for the Hamiltonian $\bar{H}$ of (3.3). This also is a circle in the ξ-plane but the centre is displaced from the origin. An example of such an orbit is seen in figure 1. This orbit is related in turn to the orbit from (2.19) for the Hamiltonian of (3.2) by the transformation of (2.36). Thus

$$|\xi''(t), k\rangle = V_2^{+}(t)\overline{|\xi'(t), k\rangle} \tag{3.13}$$

yields

$$\xi''(t) = \frac{\xi'(t)\cosh\frac{1}{2}\lambda(t) + \sinh\frac{1}{2}\lambda(t)}{\xi'(t)\sinh\frac{1}{2}\lambda(t) + \cosh\frac{1}{2}\lambda(t)} \tag{3.14}$$

where $\lambda(t) = \ln(M_0/M(t)) = 2\gamma t$ and where $\xi'(t)$ is given by (3.12). This provides an analytic solution to (2.19) with $A(t)$ and $f(t)$ given by (2.23) which, for the decaying mass are just

$$\begin{aligned} A(t) &= 2\omega_0 \cosh(2\gamma t) \\ f(t) &= -\omega_0 \sinh(2\gamma t). \end{aligned} \tag{3.15}$$

Typical orbits determined from the equation are shown in figure 2. We note that the orbits, regardless of the starting point, are attracted to the point $\xi = (1, 0)$, asymptotically approaching the unit circle. This type of behaviour was previously discussed in [16].

It is also worthwhile to calculate the variances of the field quadratures for such states. We use the physical variances in the form

$$\begin{aligned} \langle(\Delta X_1)^2\rangle &= \left(\frac{M(t)}{M_0}\right)\langle K_0 + K_1\rangle \\ \langle(\Delta X_2)^2\rangle &= \left(\frac{M_0}{M(t)}\right)\langle K_0 - K_1\rangle \end{aligned} \tag{3.16}$$

where the expectation values are with respect to the SU(1, 1) CS with ξ determined from (2.19) with (3.15). For the decaying mass

$$\begin{aligned} \langle(\Delta X_1)^2\rangle &= \mathrm{e}^{-2\gamma t}\langle K_0 + K_1\rangle \\ \langle(\Delta X_2)^2\rangle &= \mathrm{e}^{2\gamma t}\langle K_0 - K_1\rangle. \end{aligned} \tag{3.17}$$

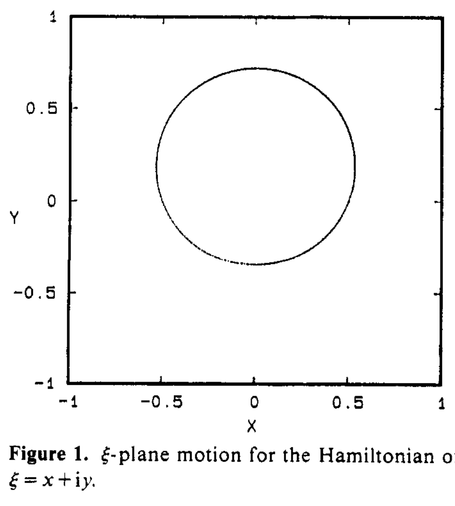


Figure 1. ξ-plane motion for the Hamiltonian of (3.3) where $\gamma = 0.5$ and $\xi(0) = 0.5$ where $\xi = x + \mathrm{i}y$.

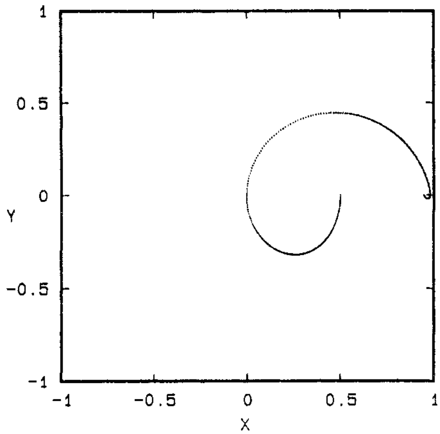


Figure 2. Motion in the ξ-plane for the Hamiltonian of (3.2) with $\gamma=0.5$ and $\xi(0)=0.5$. Because of (3.14), this is *not* the same $\xi(0)$ of figure 1.

At first sight it would appear that the variance of X_1 should approach zero as $t\to\infty$; however $\langle K_0+K_1\rangle$ approaches infinity as $\xi\to(1,0)$ in this limit so the variances in fact remain finite as shown in figure 3. What we see in this picture is the combined effects of the motion seen in figure 2 where the orbit spirals to the unit circle on loops decreasing in size and the exponential factors in (3.17). This observation is in accordance with a conjecture made in the appendix of [16]. This is in fact the same behaviour we obtain by using the Hamiltonian of (3.3) and the variance formulae of (2.34). We have checked that, in fact, the oscillation frequency ω of these variances is reduced below ω_0 in accordance with the relation $\omega=(\omega_0^2-\gamma^2)^{1/2}$. Thus we agree with Colegrave and Abdalla [1] that the only effect of the decaying mass is to alter the cavity frequency. The fact that the variances of X_1 and X_2 continuously oscillate at frequency ω rather than approach some limit (such as those of the vacuum) physically seems to imply that the fluctuations of the field are independent of its decay. In fact, the expectation values of the fields $\mathscr{E}_x(z,t)$ and $\mathscr{H}_y(z,t)$ are always zero anyway for the squeezed

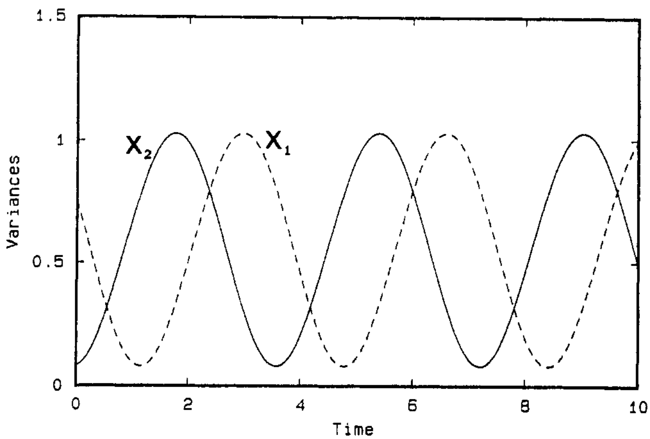


Figure 3. Variances of X_1 and X_2 for the evolution described in figure 2.

vacuum states owing to the fact that only the even photon number states compose the squeezed vacuum.

3.2. *Oscillating mass*

We now consider a mass law of the form

$$M(t)=M_0\times\begin{cases}1 & t<0\\ 1+\alpha\sin\delta t,\ |\alpha|<1 & t\geqslant 0.\end{cases} \tag{3.18}$$

This mass dependence simulates the fluctuations of the field in a cavity in resonance with a reservoir of two-level atoms. The photon population in the cavity is modified by the Rabi oscillations of the atoms in a beam traversing the cavity, where δ is then taken to be the Rabi frequency [2]. The quantal harmonic oscillator with such a time dependent mass has also previously been studied by Remaud and Hernandez [29]. They found a resonance for the case $\delta=2\omega_0$ where ω_0 is the natural frequency of the oscillator.

We shall work exclusively in the picture where the Hamiltonian is given by (2.30). For this case, for $t\geqslant 0$, we have

$$\bar{H}(t)=2\omega_0K_0+\frac{\mathrm{i}}{2}\left(\frac{\alpha\delta\cos\delta t}{1+\alpha\sin\delta t}\right)(K_+-K_-). \tag{3.19}$$

In figure 4 we present typical orbits in the ξ-plane for various δ with $\alpha=0.5$. For $\delta\ll 2\omega_0$ (where $\omega_0=1$), the orbits apparently are confined to regions where $|\xi|<1$. On the other hand, as δ is increased toward the resonance value, the orbits begin to slowly spiral out to the unit circle. The motion to the unit circle is more rapid at the resonance $\delta=2\omega_0$. As δ increases beyond the resonance, the motion appears to take on some interesting behaviour but remains well within the unit circle. For very high $\delta(\delta\sim 20)$ the motion starts to resemble that obtained in another work [30] where the SU(1, 1) CS were subjected to quasiperiodic pulsing. The patterns we obtain here resemble the patterns of [29] only for low pulsing strength.

We also calculate the variances of X_{10} and X_{20} at the resonance condition. The results are displayed in figure 5. It is clear that as $|\xi|\to 1$ a high degree of squeezing (i.e. enhanced squeezing) of the initial SU(1, 1) CS will occur.

3.3. *Damping with pulsation*

With

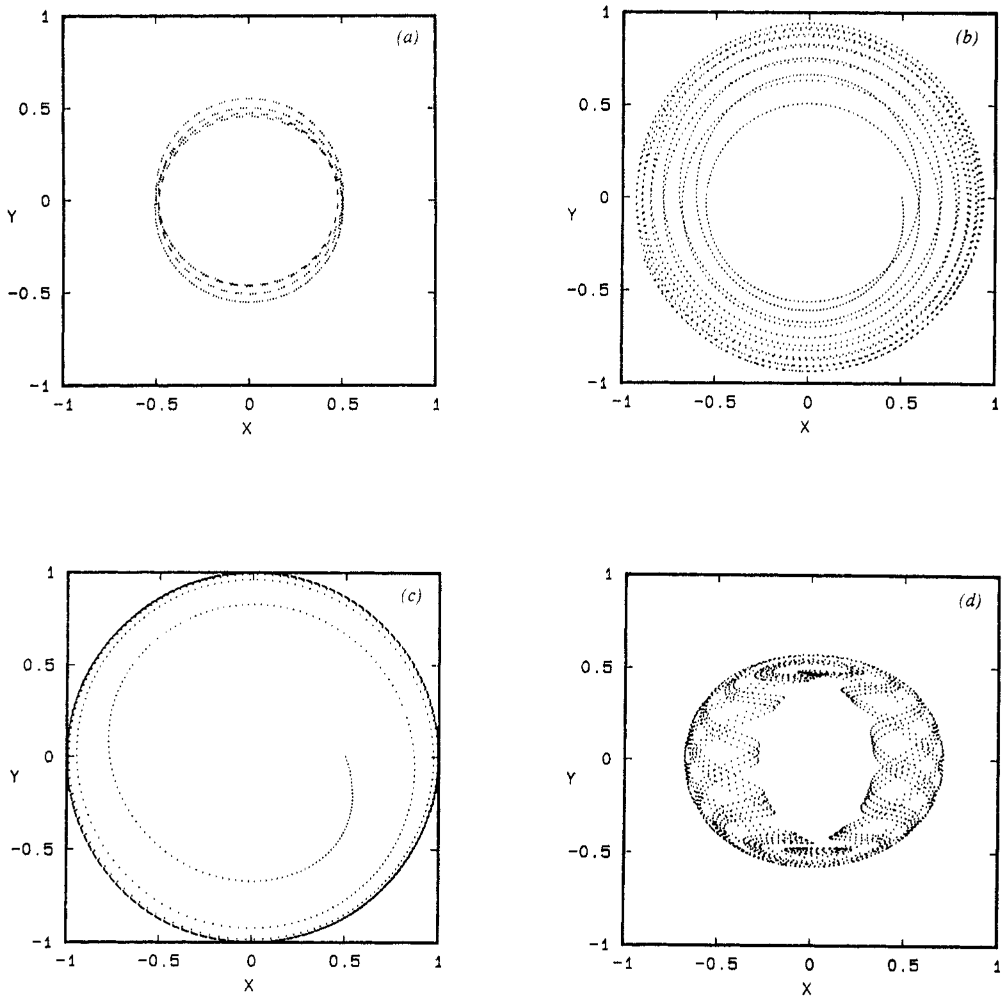
$$M(t)=M_0\exp[2(\gamma t+\mu\sin\nu t)] \tag{3.20}$$

we have

$$\bar{H}(t)=2\omega_0K_0+\frac{\mathrm{i}}{2}(2\gamma+\mu\nu\cos\nu t)(K_+-K_-). \tag{3.21}$$

The mass law above has also been shown to be relevant to the description of periodic fluctuations in a cavity in resonance with a reservoir of two-level atoms and with cavity damping [31].

In figure 6 we display typical orbits in the ξ-plane. For $\gamma=1$, $\omega_0=1$, $\mu=1$ and $\nu=0.5$, the orbit starting at $\xi(0)=0.5$ approaches the unit circle at $\xi=(0,1)$, decelerating as it does so, coming to rest at $(0,1)$ (figure 6(a)). It is easy to show that for



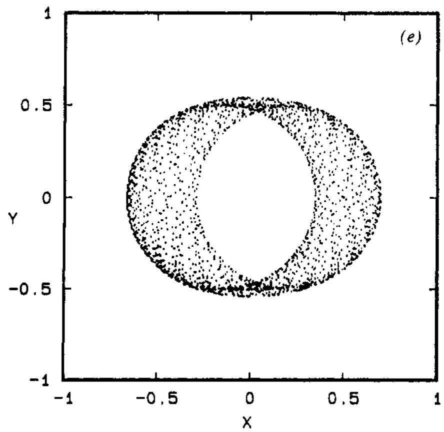


Figure 4. Motion in the ξ-plane for the pulsing mass of (3.18) for $\alpha = 0.5$, $\xi(0) = 0.5$, $\omega_0 = 1$ and (*a*) $\delta = 0.5\omega_0$, (*b*) $\delta = \omega_0$, (*c*) $\delta = 2\omega_0$ (resonance), (*d*) $\delta = 8\omega_0$, (*e*) $\delta = 20\omega_0$. 4000 points are plotted.

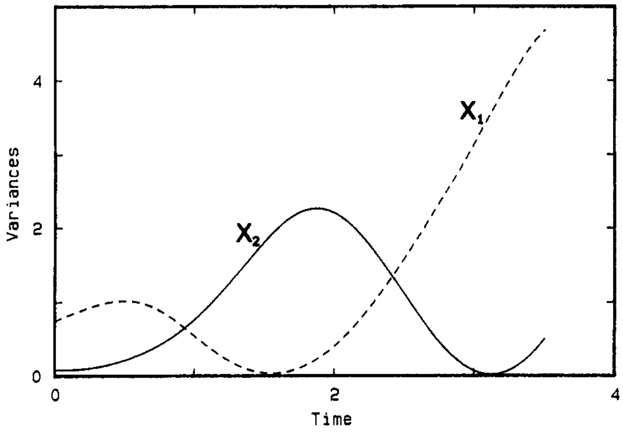


Figure 5. Variances of X_{10} and X_{20} at the resonance condition of figure 4(c).

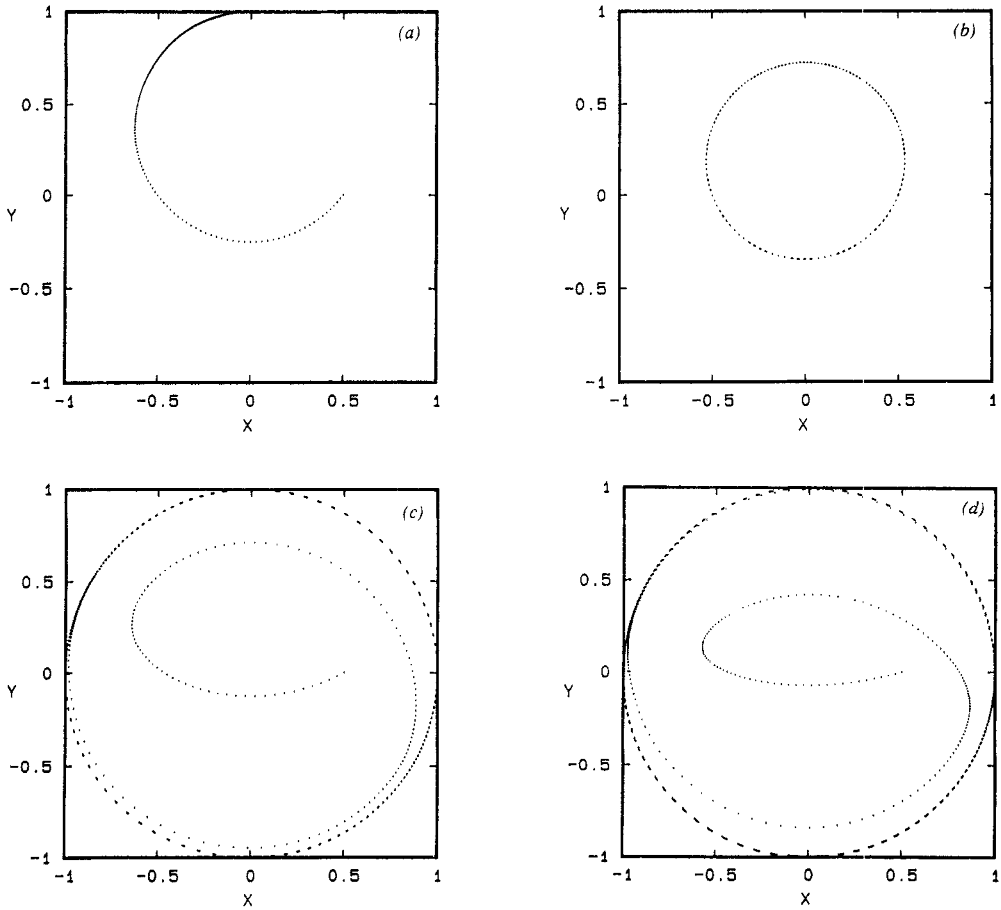


Figure 6. Motion in the ξ-plane for the damped, pulsing mass of (3.20). (a) $\gamma=1$, $\omega_0=1$, $\mu=1$, $\xi(0)=0.5$ and $\nu=0.5$; (b) same as (a) but with $\gamma=0.5$, (c) $\gamma=0.5$ and $\nu=2\omega_0$, (d) $\nu=4\omega_0$.

$A = \text{constant}$ and $f(t)$ an imaginary function of time, the unit circle is a set of stationary points for the differential equation (2.19). On the other hand for $\gamma = 0.5$ (all other parameters fixed) the motion is periodic in the ξ-plane well within the unit circle (figure 6(b)). With $\gamma = 0.5$ and $\nu = 2\omega_0$ the motion rapidly spirals out to the unit circle (figure 6(c)). The condition $\nu = 2\omega_0$ is the resonance condition described by Abdalla and Ramjit [9]. For $\nu = 4\omega_0$ the motion again tends to the unit circle but less rapidly. We do not display the associated variances since these may easily be inferred from the orbits in figure 6 and (2.35).

5. Conclusions

In this paper, we have described the evolution of SU(1, 1) CS in harmonic oscillators with time-dependent masses. The Hamiltonians driving the states are coherence preserving so the dynamics is described by the 'classical' motion in the phase space in the form of the Lobachevsky plane where the motion is determined by an inhomogeneous nonlinear differential equation (Riccati equation). The variable mass harmonic oscillator may represent a quantised electromagnetic field in a cavity where the field varies with time under the action of some reservoir. We considered three different types of variable mass: decay masses, pulsing masses and masses damped with pulsation. For the first case, the major effect is to modify the frequency of the cavity, while for the latter two a high degree of enhanced squeezing is possible as the pulsing frequency approaches the resonance value of $2\omega_0$.

Finally we point out that the approach to the time evolution we have taken here is not the only possible one. Another approach would be to use the Wei–Norman type procedures exploited by Dattoli *et al* [32] or by using a path integral approach [33] recently developed for coherence preserving Hamiltonians. The method we have used here we believe to be the most direct.

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